

Bilinear Decompositions of Products of Hardy and Lipschitz Spaces Through Wavelets

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Abstract The aim of this article is to give a complete solution to the problem of the bilinear decompositions of the products of some Hardy spaces $H^p(\mathbb{R}^n)$ and their duals in the case when $p < 1$ and near to 1, via wavelets, paraproducts and the theory of bilinear Calderón-Zygmund operators. Precisely, the authors establish the bilinear decompositions of the product spaces $H^p(\mathbb{R}^n) \times \dot{\Lambda}_\alpha(\mathbb{R}^n)$ and $H^p(\mathbb{R}^n) \times \Lambda_\alpha(\mathbb{R}^n)$, where, for all $p \in (\frac{n}{n+1}, 1)$ and $\alpha := n(\frac{1}{p} - 1)$, $H^p(\mathbb{R}^n)$ denotes the classical real Hardy space, and $\dot{\Lambda}_\alpha$ and Λ_α denote the homogeneous, respectively, the inhomogeneous Lipschitz spaces. Sharpness of these two bilinear decompositions are also proved. As an application, the authors establish some div-curl lemmas at the endpoint case.

1 Introduction

The products of functions in the Hardy space $H^1(\mathbb{R}^n)$ and its dual space $\text{BMO}(\mathbb{R}^n)$ were first studied by Bonami et al. in [6]. They proved that these products make sense as distributions, and the product space $H^1(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^n)$ has a linear decomposition of the form

$$H^1(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + H_w^\Phi(\mathbb{R}^n),$$

where $L^1(\mathbb{R}^n)$ denotes the usual Lebesgue space and $H_w^\Phi(\mathbb{R}^n)$ the weighted Orlicz-Hardy space related to the weight function $w(x) := \frac{1}{\log(e+|x|)}$ for all $x \in \mathbb{R}^n$ and the Orlicz function

$$\Phi(t) := \frac{t}{\log(e+t)} \quad \text{for all } t \in [0, \infty).$$

Later, Bonami et al. [5] proved that the product space $H^1(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^n)$ has a bilinear decomposition of the following form:

$$(1.1) \quad H^1(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + H^{\log}(\mathbb{R}^n),$$

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where $H^{\log}(\mathbb{R}^n)$ denotes the so-called Musielak-Orlicz-Hardy space, introduced in [19], related to the Musielak-Orlicz function

$$\log(x, t) := \frac{t}{\log(e + |x|) + \log(e + t)} \quad \text{for all } (x, t) \in \mathbb{R}^n \times [0, \infty).$$

It is known that the bilinear decomposition obtained in [5] essentially improves the corresponding result in [6], due to the fact that the Musielak-Orlicz-Hardy space $H^{\log}(\mathbb{R}^n)$ is a proper subspace of the weighted Orlicz-Hardy space $H_w^\Phi(\mathbb{R}^n)$ in [6]. Recall that the study of the products of functions (or distributions) in the Hardy space and its dual space has applications in many research areas such as the geometric function theory and the non-linear elasticity (see [7, 6] and their references for more details). We also refer the reader to [18, 20] for some interesting applications of the bilinear decomposition (1.1) in studying the endpoint estimates for commutators of Calderón-Zygmund operators and $\text{BMO}(\mathbb{R}^n)$ functions.

For the case p less than 1, Bonami and Feuto [3] established a series of decompositions for the products of the Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0, 1)$ and their dual spaces as follows:

$$(1.2) \quad H^p(\mathbb{R}^n) \times \Lambda_{n(\frac{1}{p}-1)}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + H^p(\mathbb{R}^n)$$

and

$$(1.3) \quad H^p(\mathbb{R}^n) \times \dot{\Lambda}_{n(\frac{1}{p}-1)}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + H_{w_\gamma}^p(\mathbb{R}^n),$$

where $H^p(\mathbb{R}^n)$ denotes the classical real Hardy space (see (2.12) below for its definition), $\dot{\Lambda}_{n(\frac{1}{p}-1)}(\mathbb{R}^n)$ and $\Lambda_{n(\frac{1}{p}-1)}(\mathbb{R}^n)$ the homogeneous, respectively, the inhomogeneous Lipschitz spaces (see (3.1) and (3.2) below for their definitions), and $H_{w_\gamma}^p(\mathbb{R}^n)$ the weighted Hardy space related to the weight function

$$(1.4) \quad w_\gamma(x) := \frac{1}{(1 + |x|)^\gamma} \quad \text{for all } x \in \mathbb{R}^n \text{ with } \gamma \in (n(1 - p), \infty).$$

All these interesting results give the decompositions of the products of Hardy spaces and their duals with exponents p less than 1. However, it should be pointed out that the decomposition obtained in (1.2) through (1.3) is not bilinear. Also, as was pointed out in [3], the range space on the right hand side of the decomposition in (1.3) is not sharp.

Motivated by the aforementioned works, in this article, we give a complete solution to the problem of bilinear decompositions of the products of Hardy spaces and their duals in the case when $p < 1$ and near to 1, via wavelets, paraproducts and the theory of bilinear Calderón-Zygmund operators. Precisely, the main result of the present article is to establish two bilinear decompositions of the following forms: for all $p \in (\frac{n}{n+1}, 1)$,

$$(1.5) \quad H^p(\mathbb{R}^n) \times \Lambda_{n(\frac{1}{p}-1)}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + H^p(\mathbb{R}^n)$$

and

$$(1.6) \quad H^p(\mathbb{R}^n) \times \dot{\Lambda}_{n(\frac{1}{p}-1)}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + H_w^p(\mathbb{R}^n),$$

where $H_w^p(\mathbb{R}^n)$ denotes the weighted Hardy space related to the weight function

$$(1.7) \quad w(x) := \frac{1}{(1 + |x|)^{n(1-p)}} \quad \text{for all } x \in \mathbb{R}^n$$

(see Theorem 3.8 below for more details). From their definitions, we immediately deduce that $H_w^p(\mathbb{R}^n) \subset H_{w_\gamma}^p(\mathbb{R}^n)$ (see Remark 3.9 below for more details). Thus, our result essentially improves the corresponding one in [3]. Also, we point out that the bilinear decompositions obtained in (1.5) and (1.6) are sharp in the sense that the range spaces, on the right hand sides of the bilinear decompositions, cannot be replaced by smaller spaces (see Remark 3.9 below for more details).

As in [5], the main idea used to obtain the bilinear decompositions (1.5) and (1.6) in this article bases on the normalization of the products of functions (or distributions) via wavelets, which was first introduced by Coifman et al. [7, 10]. More precisely, by using the theory of multiresolution analysis (MRA), we decompose the product $f \times g$ of a function (or distribution) f in the Hardy spaces and an element g in their duals into four parts $\{\Pi_i(f, g)\}_{i=1}^4$ (see (2.8) through (2.11) below for their precise definitions), where, for each $i \in \{1, 2, 3, 4\}$, the operator Π_i is bilinear and can be represented via a wavelet expansion. Thus, the idea of renormalization enables us to reduce the problem to the study of the boundedness of the four bilinear operators $\{\Pi_i\}_{i=1}^4$ in some suitable function spaces. By proving that these paraproduct operators are bilinear Calderón-Zygmund operators from [14, 15] (see Proposition 2.4 below), we obtain the boundedness of these bilinear operators in some suitable function spaces (see Propositions 3.4, 3.5 and 3.7 below for more details). Moreover, to obtain the sharp range space $H_w^p(\mathbb{R}^n)$, we also need to make full use of the wavelet coefficient characterizations of the Hardy spaces and their duals (see Theorems 3.2 and 3.3 below). We should point out that it is the use of wavelets that make us to restrict the range of exponents p to $(\frac{n}{n+1}, 1)$, since we can only obtain 0-order cancellation moment condition from the orthogonality of the wavelet basis.

As an application of the main results of this article, we study the div-curl lemma at the endpoint case. More precisely, let $p \in (\frac{n}{n+1}, 1)$ and $\alpha := n(\frac{1}{p} - 1)$, using the bilinear decomposition (1.6), we are able to prove some div-curl lemmas at the endpoint case $q = \infty$. We show that, for all $\mathbf{F} \in H^p(\mathbb{R}^n; \mathbb{R}^n)$ (the vector-valued Hardy space; see (4.1) below for its definition) with $\text{curl } \mathbf{F} \equiv 0$ in the sense of distributions and $\mathbf{G} \in \dot{\Lambda}_\alpha(\mathbb{R}^n; \mathbb{R}^n)$ (the vector-valued homogeneous Lipschitz space) with $\text{div } \mathbf{G} \equiv 0$ in the sense of distributions, it holds true that $\mathbf{F} \cdot \mathbf{G} \in H_w^p(\mathbb{R}^n)$ with w as in (1.7) and

$$(1.8) \quad \|\mathbf{F} \cdot \mathbf{G}\|_{H_w^p(\mathbb{R}^n)} \leq C \|\mathbf{F}\|_{H^p(\mathbb{R}^n; \mathbb{R}^n)} \|\mathbf{G}\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n; \mathbb{R}^n)},$$

where C is a positive constant, independent of F and G . This result essentially extends the corresponding ones in [3, 5]; see also [8, 4] for more related results on the div-curl lemma.

This article is organized as follows. Section 2 mainly deals with the renormalization of the product of two functions (or distributions). We first, in Section 2.1, recall some preliminaries on the theory of multiresolution analysis (MRA) and the wavelets arising from MRA; then, in Section 2.2, we describe the renormalization of the products of functions in $L^2(\mathbb{R}^n)$ via wavelets; in Section 3, we prove the bilinear decompositions (1.5) and

(1.6) (see Theorem 3.8 below); finally, in Section 4, we prove the div-curl lemma (1.8) (see Theorem 4.1 below).

We end this section by making some conventions on the notation. Throughout the whole article, we always let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. We use C to denote a *positive constant* that is independent of the main parameters involved but whose value may differ from line to line, and $C_{(\alpha, \dots)}$ to denote a *positive constant* depending on the indicated parameters α, \dots . *Constants with subscripts*, such as C_1 , do not change in different occurrences. If $f \leq Cg$, we then write $f \lesssim g$ and, if $f \lesssim g \lesssim f$, we then write $f \sim g$. For any $\lambda \in (0, \infty)$, $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, let

$$B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\} \text{ and } \lambda B := B(x, \lambda r).$$

Also, for any set $E \subset \mathbb{R}^n$, χ_E denotes its *characteristic function*. We let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class on \mathbb{R}^n and $\mathcal{S}'(\mathbb{R}^n)$ the space of all tempered distributions. The notation $f * g$ always denotes the *convolution* of two functions f and g , or a Schwartz function f and a tempered distribution g . For any function h , we use \widehat{h} to denote its *Fourier transform*. Also, for any $x := (x_1, \dots, x_n) \in \mathbb{R}^n$ and multi-index $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we let $\partial_x^\alpha := (\frac{\partial}{\partial x_1})^{\alpha_1} \dots (\frac{\partial}{\partial x_n})^{\alpha_n}$ and $x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}$.

2 Renormalization of Products via Wavelets

The main purpose of this section is to give a renormalization of the pointwise product of any two functions in $L^2(\mathbb{R}^n)$ via wavelets. This kind of renormalization constitutes the basis of our method to obtain the bilinear decompositions for the products of Hardy and Lipschitz or BMO spaces. We first recall in Section 2.1 some preliminaries on the homogeneous multiresolution analysis (MRA) and the wavelets arising from MRA; then, in Section 2.2, we renormalize the products of functions in $L^2(\mathbb{R}^n)$ into four bilinear operators through wavelets.

2.1 MRA and Wavelets

We begin this subsection by recalling the definition of the (homogeneous) multiresolution analysis on \mathbb{R} ; see, for example, [21, 16] for more details.

Definition 2.1. Let $\{V_j\}_{j \in \mathbb{Z}}$ be an increasing sequence of closed subspaces in $L^2(\mathbb{R})$. Then $\{V_j\}_{j \in \mathbb{Z}}$ is called a *multiresolution analysis* (for short, MRA) on \mathbb{R} if it has the following properties:

- (i) $\bigcap_{j \in \mathbb{Z}} V_j = \{\theta\}$ and $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R})$, where θ is the zero element of $L^2(\mathbb{R})$;
- (ii) for any $j \in \mathbb{Z}$ and $f \in L^2(\mathbb{R})$, $f(\cdot) \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$;
- (iii) for any $f \in L^2(\mathbb{R})$ and $k \in \mathbb{Z}$, $f(\cdot) \in V_0$ if and only if $f(\cdot - k) \in V_0$;
- (iv) there exists a function $\phi \in L^2(\mathbb{R})$ (called a *scaling function* or *father wavelet*) such that $\{\phi_k(\cdot)\}_{k \in \mathbb{Z}} := \{\phi(\cdot - k)\}_{k \in \mathbb{Z}}$ is an orthonormal basis of V_0 .

For all $j \in \mathbb{Z}$, let $W_j := V_{j+1} \ominus V_j$ be the *orthogonal complement* of V_j in V_{j+1} . Then it is easy to see that $V_{j+1} = \bigoplus_{i=-\infty}^j W_i$ and $L^2(\mathbb{R}) = \bigoplus_{i=-\infty}^{\infty} W_i$, here and hereafter, \bigoplus denotes the *orthogonal direct sum* in $L^2(\mathbb{R})$.

By Definition 2.1, we know that, for all $j \in \mathbb{Z}$, $\{2^{j/2}\phi(2^j \cdot -k)\}_{k \in \mathbb{Z}}$ is an orthogonal basis of V_j . Also, from the classical theory of MRA (see, for example, [9, 21, 16]), it follows that there exists a function $\psi \in L^2(\mathbb{R})$, called a *wavelet* or *mother wavelet*, such that, for all $j \in \mathbb{Z}$, $\{2^{j/2}\psi(2^j \cdot -k)\}_{k \in \mathbb{Z}}$ is an orthogonal basis of W_j . Moreover, for any $N \in \mathbb{N}$, it is known that one can construct the father and the mother wavelets $\phi, \psi \in C^N(\mathbb{R})$ (the set of all functions with continuous derivatives up to order N) with compact supports such that $\widehat{\phi}(0) = (2\pi)^{-1/2}$, where $\widehat{\phi}$ denotes the *Fourier transform* of ϕ , and, for all $l \in \{0, \dots, N\}$,

$$(2.1) \quad \int_{\mathbb{R}} x^l \psi(x) dx = 0$$

(see, for example, [9]). Recall that there does not exist a wavelet basis in $L^2(\mathbb{R})$ whose elements are both infinitely differentiable and have compact supports (see, for example, [16, Theorem 3.8]).

Following [5], throughout the whole article, we always assume that

$$(2.2) \quad \text{supp } \phi, \text{ supp } \psi \subset 1/2 + m(-1/2, 1/2),$$

where $1/2 + m(-1/2, 1/2)$ denotes the interval obtained from $(0, 1)$ via a dilation by m centered at $1/2$, namely, $x \in 1/2 + m(-1/2, 1/2)$ if and only if $|x - 1/2| < m/2$. Here m is a positive constant that is independent of the main parameters considered in the whole article.

The extension of the above considerations from one dimension to any n -dimension follows from a standard procedure of tensor products. More precisely, let $\{\widetilde{V}_j\}_{j \in \mathbb{Z}}$ be a multiresolution analysis on \mathbb{R} . For all $j \in \mathbb{Z}$ and $n \in \mathbb{N}$, let

$$(2.3) \quad V_j := \overbrace{\widetilde{V}_j \otimes \cdots \otimes \widetilde{V}_j}^{n \text{ times}}$$

be the n -fold *tensor product* of \widetilde{V}_j and $W_j := V_{j+1} \ominus V_j$ be the *orthogonal complement* of V_j in V_{j+1} . Then it is easy to see that

$$(2.4) \quad V_{j+1} = \bigoplus_{i=-\infty}^j W_i \quad \text{and} \quad L^2(\mathbb{R}^n) = \bigoplus_{i=-\infty}^{\infty} W_i.$$

Moreover, let $E := \{0, 1\}^n \setminus \{(0, \dots, 0)\}$. For all $\lambda := (\lambda_1, \dots, \lambda_n) \in E$ and $x := (x_1, \dots, x_n) \in \mathbb{R}^n$, let

$$(2.5) \quad \psi^\lambda(x) := \phi^{\lambda_1}(x_1) \cdots \phi^{\lambda_n}(x_n),$$

where, for all $i \in \{1, \dots, n\}$,

$$\phi^{\lambda_i}(x_i) := \begin{cases} \phi(x_i) & \text{when } \lambda_i = 0, \\ \psi(x_i) & \text{when } \lambda_i = 1. \end{cases}$$

For all $j \in \mathbb{Z}$ and $k := \{k_1, \dots, k_n\} \in \mathbb{Z}^n$, let

$$I_{j,k} := \{x \in \mathbb{R}^n : k_i \leq 2^j x_i < k_i + 1 \text{ for all } i \in \{1, \dots, n\}\}$$

be the *dyadic cube* with the lower left-corner $x_{j,k} := 2^{-j}k$. For all $x := (x_1, \dots, x_n) \in \mathbb{R}^n$, let

$$\phi_{I_{0,0}}(x) := \phi(x_1) \cdots \phi(x_n).$$

From Definition 2.1 and (2.4), we deduce that, for all $j \in \mathbb{Z}$,

$$\{\phi_{I_{j,k}}(\cdot)\}_{k \in \mathbb{Z}^n} := \left\{2^{jn/2} \phi_{I_{0,0}}(2^j \cdot -k)\right\}_{k \in \mathbb{Z}^n}$$

and

$$\left\{\psi_{I_{j,k}}^\lambda(\cdot)\right\}_{\lambda \in E, k \in \mathbb{Z}^n} := \left\{2^{jn/2} \psi^\lambda(2^j \cdot -k)\right\}_{\lambda \in E, k \in \mathbb{Z}^n}$$

form orthogonal bases of V_j , respectively, W_j . In what follows, to simplify the notation, we often write I instead $I_{j,k}$ when there exists no confusion. By (2.2), we conclude that, for all dyadic cubes I ,

$$(2.6) \quad \text{supp } \phi_I, \text{ supp } \psi_I \subset mI,$$

where mI denotes the m dilation of I with the same center as I . Thus, for any $f \in L^2(\mathbb{R}^n)$, we have the following wavelet expansion

$$f = \sum_{I \in \mathcal{D}} \sum_{\lambda \in E} \langle f, \psi_I^\lambda \rangle \psi_I^\lambda$$

holds true in $L^2(\mathbb{R}^n)$, here and hereafter, \mathcal{D} always denotes the set of all the *dyadic cubes* in \mathbb{R}^n and, unless otherwise stated, $\langle \cdot, \cdot \rangle$ the inner product in $L^2(\mathbb{R}^n)$.

2.2 Renormalization of Pointwise Products in $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$

Let f and g be two functions in $L^2(\mathbb{R}^n)$. In this subsection, we renormalize the product fg by using the wavelet basis in Section 2.1. Before continuing, we first point out that this method of renormalization was first introduced by Coifman et al. [7] and Dobyinsky [10].

For all $j \in \mathbb{Z}$, let V_j and W_j be as in Section 2.1, and P_j and Q_j be the *orthogonal projections* of $L^2(\mathbb{R}^n)$ onto V_j , respectively, onto W_j . Dobyinsky [10] proved that, for all $f, g \in L^2(\mathbb{R}^n)$,

$$(2.7) \quad fg = \sum_{j \in \mathbb{Z}} (P_j f)(Q_j g) + \sum_{j \in \mathbb{Z}} (Q_j f)(P_j g) + \sum_{j \in \mathbb{Z}} (Q_j f)(Q_j g)$$

holds true in $L^1(\mathbb{R}^n)$. Based on (2.7) and following [10, 5], we introduce four bilinear operators via the wavelet basis as follows. For all $f, g \in L^2(\mathbb{R}^n)$, define

$$(2.8) \quad \Pi_1(f, g) := \sum_{\substack{I, I' \in \mathcal{D} \\ |I|=|I'|}} \sum_{\lambda \in E} \langle f, \phi_I \rangle \langle g, \psi_{I'}^\lambda \rangle \phi_I \psi_{I'}^\lambda,$$

$$(2.9) \quad \Pi_2(f, g) := \sum_{\substack{I, I' \in \mathcal{D} \\ |I|=|I'|}} \sum_{\lambda \in E} \langle f, \psi_I^\lambda \rangle \langle g, \phi_{I'} \rangle \psi_I^\lambda \phi_{I'},$$

$$(2.10) \quad \Pi_3(f, g) := \sum_{\substack{I, I' \in \mathcal{D} \\ |I|=|I'|}} \sum_{\substack{\lambda, \lambda' \in E \\ (I, \lambda) \neq (I', \lambda')}} \langle f, \psi_I^\lambda \rangle \langle g, \psi_{I'}^{\lambda'} \rangle \psi_I^\lambda \psi_{I'}^{\lambda'}$$

and

$$(2.11) \quad \Pi_4(f, g) := \sum_{I \in \mathcal{D}} \sum_{\lambda \in E} \langle f, \psi_I^\lambda \rangle \langle g, \psi_I^\lambda \rangle \left(\psi_I^\lambda \right)^2.$$

We point out that, although the bilinear operators are only defined in $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, they can be extended to the products of more general function spaces in the case even when f and g are distributions and have wavelet expansions (see, for example, [21, 16] for the Hardy spaces or [26, 27] for more general scales of Besov and Triebel-Lizorkin spaces). In this case, the symbol $\langle \cdot, \cdot \rangle$ may denote the dual pair between distribution and the associated wavelet function.

From (2.7) and the facts that $\{\phi_{I_{j,k}}\}_{k \in \mathbb{Z}^n}$ and $\{\psi_{I_{j,k}}^\lambda\}_{\lambda \in E, k \in \mathbb{Z}^n}$ constitute orthogonal bases of V_j , respectively, W_j , it follows that

$$fg = \sum_{i=1}^4 \Pi_i(f, g)$$

holds true in $L^1(\mathbb{R}^n)$. As is in [10, 5], we always let $T := \sum_{i=1}^3 \Pi_i$ and $S := \Pi_4$. It will be seen that both the operators S and T inherit some properties of the factors f and g . In particular, the operator T , which preserves the cancellation properties of the Hardy spaces, is usually called the *renormalization of the product* fg .

The remainder of this section is devoted to the study of some basic bounded properties of the bilinear operators $\{\Pi_i\}_{i=1}^4$. To this end, we first recall the definition of the Hardy space $H^p(\mathbb{R}^n)$ for $p \in (0, \infty)$ from [11]. For any $m \in \mathbb{N}$, let

$$\mathcal{S}_m(\mathbb{R}^n) := \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \leq m+1}} (1 + |x|)^{(m+2)(n+1)} |\partial_x^\alpha \varphi(x)| \leq 1 \right\}.$$

For any $m \in \mathbb{N}$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define the *non-tangential grand maximal function* $f_m^*(x)$ of f by setting

$$f_m^*(x) := \sup_{\varphi \in \mathcal{S}_m(\mathbb{R}^n)} \sup_{|y-x| < t, t \in (0, \infty)} |f * \varphi_t(y)|,$$

here and hereafter, for all $t \in (0, \infty)$, $\varphi_t(\cdot) := \frac{1}{t^n} \varphi(\frac{\cdot}{t})$ and we always remove the subscript m whenever $m > \lfloor n(\frac{1}{p} - 1) \rfloor$. Recall that, as usual, the symbol $\lfloor s \rfloor$ for any $s \in \mathbb{R}$ denotes the largest integer smaller than or equal to s .

Then, for all $p \in (0, \infty)$, the *Hardy space* $H^p(\mathbb{R}^n)$ is defined to be the set

$$(2.12) \quad H^p(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{H^p(\mathbb{R}^n)} := \|f^*\|_{L^p(\mathbb{R}^n)} < \infty\}.$$

For more properties of the Hardy spaces and their various characterizations, we refer the reader to [11, 24].

The following theorem, on the boundedness of the operators S and T , was first proved by Dobyinsky [10].

Theorem 2.2 ([10]). (i) *The operator S is a bilinear operator bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.*

(ii) *The operator T is a bilinear operator bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$.*

Remark 2.3. From Theorem 2.2, we deduce that, for any $f, g \in L^2(\mathbb{R}^n)$,

$$fg = S(f, g) + T(f, g) \in L^1(\mathbb{R}^n) + H^1(\mathbb{R}^n).$$

This implies the following bilinear decomposition of the product space $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$,

$$L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + H^1(\mathbb{R}^n).$$

Now we show that the four bilinear operators $\{\Pi_i\}_{i=1}^4$, defined as in (2.8) through (2.11), are bilinear Calderón-Zygmund operators. This fact is needed when we discuss the boundedness properties of these operators later in this article. Recall that, in [14, 15], a bilinear operator T is called a *bilinear Calderón-Zygmund operator* if it satisfies the following two conditions:

- (i) there exist $p_0, q_0 \in (1, \infty)$ and $r_0 \in [1, \infty)$ satisfying $\frac{1}{r_0} = \frac{1}{p_0} + \frac{1}{q_0}$ such that T can be extended to a bounded bilinear operator from $L^{p_0}(\mathbb{R}^n) \times L^{q_0}(\mathbb{R}^n)$ to $L^{r_0}(\mathbb{R}^n)$;
- (ii) the associated kernel function K of T satisfies the following size and the following regularity conditions: there exists a positive constant C such that, for all $(x, y, z) \in (\mathbb{R}^n)^3 \setminus \Omega$,

$$|K(x, y, z)| \leq C \frac{1}{(|x - y| + |x - z| + |y - z|)^{2n}}$$

and there exist positive constants $\epsilon \in (0, 1]$ and C such that, for all $(x, y, z), (x', y, z) \in (\mathbb{R}^n)^3 \setminus \Omega$ satisfying $|x - x'| \leq \frac{1}{2} \max\{|x - y|, |y - z|\}$,

$$(2.13) \quad |K(x, y, z) - K(x', y, z)| \leq C \frac{|x - x'|^\epsilon}{(|x - y| + |x - z| + |y - z|)^{2n+\epsilon}},$$

where $(\mathbb{R}^n)^3 := \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ and $\Omega := \{(x, y, z) \in (\mathbb{R}^n)^3 : x = y = z\}$ denotes the *diagonal set* in $(\mathbb{R}^n)^3$. Moreover, for all $(x, y, z) \in (\mathbb{R}^n)^3 \setminus \Omega$, let $K_1(x, y, z) := K(y, x, z)$ and $K_2(x, y, z) := K(z, y, x)$. Then K_1 and K_2 also satisfy (2.13).

Proposition 2.4. *The bilinear operators $\{\Pi_i\}_{i=1}^4$, defined as in (2.8) through (2.11), are bilinear Calderón-Zygmund operators with $\epsilon = 1$.*

Proof. Without loss of generality, we may only consider the bilinear operator Π_1 . The proofs of the bilinear operators $\{\Pi_i\}_{i=2}^4$ are similar, the details being omitted. By (2.8) and (2.6), we first write

$$(2.14) \quad \begin{aligned} \Pi_1(f, g) &= \sum_{k' \in (-m, m]^n} \sum_{\lambda \in E} \left[\sum_{I \in \mathcal{D}} |I|^{-1/2} \langle f, \phi_I \rangle \langle g, \psi_{I+\ell_I k'}^\lambda \rangle |I|^{1/2} \phi_I \psi_{I+\ell_I k'}^\lambda \right] \\ &=: \sum_{k' \in (-m, m]^n} \sum_{\lambda \in E} \left[\tilde{\Pi}_{1, k', \lambda}(f, g) \right], \end{aligned}$$

where, for each $I := I_{j, k} \in \mathcal{D}$, $\ell_I := 2^{-j}$ denotes its side length, and we assume that the father and the mother wavelets $\phi, \psi \in C^N(\mathbb{R})$ with some fixed $N \in (1, \infty)$. This, together with (2.6), immediately implies that, for any $M \in \mathbb{N} \cap (n, \infty)$, multi-indices γ satisfying $|\gamma| \leq N$ and $x \in \mathbb{R}^n$,

$$(2.15) \quad \left| \partial^\gamma (|I|^{1/2} \phi_I \psi_{I+\ell_I k'}^\lambda)(x) \right| \leq C_{(\gamma, N, M)} \frac{2^{jn/2} 2^{j|\gamma|}}{(1 + 2^j |x - 2^{-j} k|)^M},$$

where $C_{(\gamma, N, M)}$ is a positive constant depending on γ , N and M , but independent of j , k , k' and x . Moreover, from $\psi_{I+\ell_I k'}^\lambda \in W_j$ and $\phi_I \in V_j$, we deduce that

$$(2.16) \quad \int_{\mathbb{R}^n} (|I|^{1/2} \phi_I \psi_{I+\ell_I k'}^\lambda)(x) dx = 0.$$

This, combined with (2.15), shows $\{|I|^{1/2} \phi_I \psi_{I+\ell_I k'}^\lambda\}_{I \in \mathcal{D}}$ is a family of smooth $(N, M, 0)$ -molecules (see [12, p. 56, (3.3) through (3.6)] or [2, (3) and (5)] for their definitions). By using a similar calculation, we also find that $\{\psi_{I+\ell_I k'}^\lambda\}_{I \in \mathcal{D}}$ and $\{\phi_I \psi_{I+\ell_I k'}^\lambda\}_{I \in \mathcal{D}}$ satisfy both (2.15) and (2.16), and hence are families of smooth $(N, M, 0)$ -molecules. Thus, we conclude that $\tilde{\Pi}_{1, k', \lambda}(f, g)$ is a typical *dyadic paraproduct* of the following form:

$$\Pi(f, g) := \sum_{I \in \mathcal{D}} |I|^{-1/2} \langle f, \phi_I^1 \rangle \langle g, \phi_I^2 \rangle \phi_I^3,$$

where $\{\phi_I^j\}_{I \in \mathcal{D}}$, for any $j \in \{1, 2, 3\}$, is a family of smooth molecules. By [2, Theorem 4.1], we find out that, for any $k' \in (-m, m]^n$ and $\lambda \in E$, the bilinear operator $\tilde{\Pi}_{1, k', \lambda}$ is a bilinear Calderón-Zygmund operator with $\epsilon = 1$, which, together with (2.14), implies that the same conclusion holds true for Π_1 . This finishes the proof of Proposition 2.4. \square

Remark 2.5. Using Proposition 2.4 and the boundedness properties of bilinear Calderón-Zygmund operators (see, for example, [2, Theorem 4.2]), we immediately know that, for any $i \in \{1, 2, 3, 4\}$ and $p, q \in (1, \infty)$, Π_i is bounded from the product space $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ with $\frac{1}{r} := \frac{1}{p} + \frac{1}{q}$.

3 Products of Functions in Hardy and Lipschitz Spaces

In this section, we study the bilinear decompositions of the products of functions (or distributions) in Hardy and Lipschitz spaces.

For all $\alpha \in (0, 1]$ and f being continuous, let

$$\|f\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)} := \sup_{\substack{x, y \in \mathbb{R}^n \\ x \neq y}} \left\{ \frac{|f(x) - f(y)|}{|x - y|^\alpha} \right\} \quad \text{and} \quad \|f\|_{\Lambda_\alpha(\mathbb{R}^n)} := \|f\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)} + \|f\|_{L^\infty(\mathbb{R}^n)}.$$

Then the *homogeneous* and the *inhomogeneous Lipschitz spaces* are defined as follows:

$$(3.1) \quad \dot{\Lambda}_\alpha(\mathbb{R}^n) := \{f \text{ is continuous in } \mathbb{R}^n : \|f\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)} < \infty\},$$

respectively,

$$(3.2) \quad \Lambda_\alpha(\mathbb{R}^n) := \{f \text{ is continuous in } \mathbb{R}^n : \|f\|_{\Lambda_\alpha(\mathbb{R}^n)} < \infty\}.$$

By their definitions, it is easy to see that $\Lambda_\alpha(\mathbb{R}^n) = [\dot{\Lambda}_\alpha(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)]$.

It is well known that the homogeneous Lipschitz space $\dot{\Lambda}_\alpha(\mathbb{R}^n)$ can be characterized by the mean oscillation as follows.

Theorem 3.1 ([22, 17]). *Let $\alpha \in [0, 1]$ and $q \in [1, \infty)$. For all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, define*

$$\|f\|_{\text{BMO}_{\alpha, q}(\mathbb{R}^n)} := \sup_{\substack{x \in \mathbb{R}^n \\ r \in (0, \infty)}} \left\{ \frac{1}{|B(x, r)|^{1+\frac{\alpha}{n}}} \int_{B(x, r)} |f(y) - f_{B(x, r)}|^q dy \right\}^{\frac{1}{q}}$$

and $\text{BMO}_{\alpha, q}(\mathbb{R}^n) := \{f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_{\text{BMO}_{\alpha, q}(\mathbb{R}^n)} < \infty\}$, where, for any ball B , $f_B := \frac{1}{|B|} \int_B f(x) dx$ denotes the mean of f over B . Then, for all $\alpha \in (0, 1]$ and $q \in [1, \infty)$, $\dot{\Lambda}_\alpha(\mathbb{R}^n) = \text{BMO}_{\alpha, q}(\mathbb{R}^n)$ with equivalent norms.

Recall that, if $\alpha \equiv 0$, then the space $\text{BMO}_{\alpha, q}(\mathbb{R}^n)$ is just the well known space $\text{BMO}(\mathbb{R}^n)$, which is identified as the dual space of the Hardy space $H^1(\mathbb{R}^n)$. Also, since the space $\text{BMO}_{\alpha, q}(\mathbb{R}^n)$ is invariant under the change of q , we usually remove the subscript q when there exists no confusion.

The following result characterizes the homogeneous Lipschitz space $\dot{\Lambda}_\alpha(\mathbb{R}^n)$ in terms of the wavelet coefficients, which plays an important role in what follows.

Theorem 3.2 ([1]). *Let $\alpha \in [0, 1]$, $f \in \text{BMO}_\alpha(\mathbb{R}^n)$ and $\{\psi_I^\lambda\}_{I \in \mathcal{D}, \lambda \in E}$ be a family of mother wavelets. Then the wavelet coefficients $\{s_{I, \lambda}\}_{I \in \mathcal{D}, \lambda \in E} := \{\langle f, \psi_I^\lambda \rangle\}_{I \in \mathcal{D}, \lambda \in E}$ of f satisfy the following Carleson type condition, namely, there exists a positive constant C , independent of f , such that*

$$\|\{s_{I, \lambda}\}_{I \in \mathcal{D}, \lambda \in E}\|_{\mathcal{C}_\alpha(\mathbb{R}^n)} := \sup_{I \in \mathcal{D}} \left\{ \frac{1}{|I|^{\frac{2\alpha}{n}+1}} \sum_{\lambda \in E} \sum_{\substack{J \in \mathcal{D} \\ J \subset I}} |s_{J, \lambda}|^2 \right\}^{\frac{1}{2}} \leq C \|f\|_{\text{BMO}_\alpha(\mathbb{R}^n)}.$$

For the wavelet characterization of the Hardy space $H^p(\mathbb{R}^n)$ defined as in (2.12) with $p \in (0, \infty)$, let $\psi \in C^N(\mathbb{R})$ be a mother wavelet satisfying (2.1) and (2.2) with N sufficiently large. Let $\{\psi_I^\lambda\}_{I \in \mathcal{D}, \lambda \in E}$ be a family of the bases generated by ψ . Recall that, for all $p \in (0, \infty)$, $H^p(\mathbb{R}^n)$ coincides with the homogeneous Triebel-Lizorkin space $\dot{F}_{p,2}^0(\mathbb{R}^n)$ with equivalent quasi-norms (see, for example, [25, p. 238, Definition 2] for the definition of the homogeneous Triebel-Lizorkin space $\dot{F}_{p,2}^0(\mathbb{R}^n)$ and [25, p. 244, Theorem] for the statement of this fact). Thus, by the wavelet coefficient characterization of $\dot{F}_{p,2}^0(\mathbb{R}^n)$ (see [12, Theorem 2.2]), we know that there exists a positive constant \tilde{C} such that, for all $f \in H^p(\mathbb{R}^n)$,

$$(3.3) \quad \begin{aligned} \|f\|_{H^p(\mathbb{R}^n)} &= \tilde{C} \left\| \left\{ \sum_{\substack{I \in \mathcal{D} \\ \lambda \in E}} \left[|\langle f, \psi_I^\lambda \rangle| |I|^{-\frac{1}{2}} \chi_I(\cdot) \right]^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \\ &=: \tilde{C} \left\| \{\langle f, \psi_I^\lambda \rangle\}_{I \in \mathcal{D}, \lambda \in E} \right\|_{\dot{f}_{p,2}^0(\mathbb{R}^n)}, \end{aligned}$$

where $\dot{f}_{p,2}^0(\mathbb{R}^n)$ denotes the corresponding *homogeneous Triebel-Lizorkin sequence space* (see [12, p. 38] for its definition).

Moreover, if the element of the Hardy space $H^p(\mathbb{R}^n)$ has a finite wavelet expansion, we can obtain some further properties on the atomic decomposition of $H^p(\mathbb{R}^n)$. Recall that, for all $p \in (0, 1]$, a function $a \in L^2(\mathbb{R}^n)$ is called an $H^p(\mathbb{R}^n)$ -atom if there exists a cube I such that $\text{supp } a \subset I$, $\|a\|_{L^2(\mathbb{R}^n)} \leq |I|^{\frac{1}{2} - \frac{1}{p}}$ and, for any multi-index $\alpha := (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}_+)^n$ satisfying $|\alpha| \leq \lfloor n(\frac{1}{p} - 1) \rfloor$,

$$\int_{\mathbb{R}^n} x^\alpha a(x) dx = 0,$$

where, for any $x := (x_1, \dots, x_n) \in \mathbb{R}^n$, $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

The following theorem can be proved by a way similar to that used in the proof of [16, Theorem 5.12], the details being omitted.

Theorem 3.3. *Let $p \in (0, 1]$ and $f \in H^p(\mathbb{R}^n)$ have a finite wavelet expansion, namely,*

$$(3.4) \quad f = \sum_{I \in \mathcal{D}} \sum_{\lambda \in E} \langle f, \psi_I^\lambda \rangle \psi_I^\lambda,$$

where the coefficients $\langle f, \psi_I^\lambda \rangle \neq 0$ only for a finite number of $(I, \lambda) \in \mathcal{D} \times E$. Then f has a finite atomic decomposition satisfying $f = \sum_{l=1}^L \mu_l a_l$, with $L \in \mathbb{N}$, and there exists a positive constant C , independent of $\{\mu_l\}_{l=1}^L$, $\{a_l\}_{l=1}^L$ and f , such that

$$\left\{ \sum_{l=1}^L |\mu_l|^p \right\}^{\frac{1}{p}} \leq C \|f\|_{H^p(\mathbb{R}^n)},$$

where, for all $l \in \{1, \dots, L\}$, a_l is an $H^p(\mathbb{R}^n)$ -atom associated with some dyadic cube R_l , which can be written into the following form

$$(3.5) \quad a_l = \sum_{\substack{I \in \mathcal{D} \\ I \subset R_l}} \sum_{\lambda \in E} c_{I, \lambda, l} \psi_I^\lambda$$

with $\{c_{I, \lambda, l}\}_{I \subset R_l, \lambda \in E, l \in \{1, \dots, L\}}$ being positive constants independent of $\{a_l\}_{l=1}^L$. Moreover, for each $l \in \{1, \dots, L\}$, the wavelet expansion of a_l in (3.5) is also finite which is extracted from that of f in (3.4).

Now, we consider the product of the Hardy and the Lipschitz spaces. Let $f \in H^p(\mathbb{R}^n)$ and $g \in \dot{\Lambda}_\alpha(\mathbb{R}^n)$ with $\alpha = n(\frac{1}{p} - 1)$. Observe that f may be a distribution. Thus, we cannot define the product of f and g directly by pointwise multiplication. However, as was pointed out in [6, 3], we can define the product as the distribution in the following way. For any $p \in (0, 1)$ and $\alpha \in (0, 1)$ with $\alpha = n(\frac{1}{p} - 1)$, let $f \in H^p(\mathbb{R}^n)$ and $g \in \dot{\Lambda}_\alpha(\mathbb{R}^n)$. The product $f \times g$ is defined by setting, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$(3.6) \quad \langle f \times g, \phi \rangle := \langle \phi g, f \rangle,$$

where the last bracket denotes the dual pair between $\dot{\Lambda}_{n(\frac{1}{p}-1)}(\mathbb{R}^n)$ and $H^p(\mathbb{R}^n)$.

Recall that Nakai and Yabuta [23, Theorem 1] proved that every $\phi \in \mathcal{S}(\mathbb{R}^n)$ is a pointwise multiplier of the homogeneous Lipschitz space $\dot{\Lambda}_\alpha(\mathbb{R}^n)$ for all $\alpha \in (0, 1]$, namely, for all $g \in \dot{\Lambda}_\alpha(\mathbb{R}^n)$, $\phi g \in \dot{\Lambda}_\alpha(\mathbb{R}^n)$. This implies that the definition in (3.6) is meaningful.

If $f \in H^p(\mathbb{R}^n)$ and $g \in \Lambda_\alpha(\mathbb{R}^n)$, where $p \in (0, 1)$ and $\alpha = n(\frac{1}{p} - 1)$, then, similar to (3.6), we can define the product $f \times g$ by setting, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$(3.7) \quad \langle f \times g, \phi \rangle := \langle \phi g, f \rangle,$$

where the last bracket denotes also the dual pair between $\dot{\Lambda}_{n(\frac{1}{p}-1)}(\mathbb{R}^n)$ and $H^p(\mathbb{R}^n)$. This definition is well defined because of the facts that $\Lambda_{n(\frac{1}{p}-1)}(\mathbb{R}^n) \subset \dot{\Lambda}_{n(\frac{1}{p}-1)}(\mathbb{R}^n)$ and that every $\phi \in \mathcal{S}(\mathbb{R}^n)$ is also a pointwise multiplier of the inhomogeneous Lipschitz space $\Lambda_\alpha(\mathbb{R}^n)$ for all $\alpha \in (0, 1]$ (see [23, Theorem 2]).

To give the desired bilinear decomposition of the product $f \times g$, we first consider the boundedness of the bilinear operators $\{\Pi_i\}_{i=1}^4$ on the products of the Hardy and the Lipschitz spaces.

Proposition 3.4. *Let $p \in (\frac{n}{n+1}, 1)$ and $\alpha = n(\frac{1}{p} - 1)$. Then the bilinear operator Π_1 , defined as in (2.8), is bounded*

$$(3.8) \quad \text{from } H^p(\mathbb{R}^n) \times \Lambda_\alpha(\mathbb{R}^n) \text{ to } H^p(\mathbb{R}^n)$$

and

$$(3.9) \quad \text{from } H^p(\mathbb{R}^n) \times \dot{\Lambda}_\alpha(\mathbb{R}^n) \text{ to } H^1(\mathbb{R}^n).$$

Proof. We first prove (3.8). For any $g \in L^\infty(\mathbb{R}^n)$, define the linear operator Π_g by setting, for any $f \in H^p(\mathbb{R}^n)$, $\Pi_g(f) := \Pi_1(f, g)$. Since Π_1 is a bilinear Calderón-Zygmund operator with $\epsilon = 1$ (see Proposition 2.4), we know that the kernel function K_g of the operator Π_g satisfies that, for all $x, y \in \mathbb{R}^n$ with $x \neq y$, $|K_g(x, y)| \lesssim \|g\|_{L^\infty(\mathbb{R}^n)} \frac{1}{|x-y|^n}$ and, for all $x, x', y \in \mathbb{R}^n$ satisfying $x \neq y \neq x'$ and $|x - x'| \leq |x - y|/2$,

$$|K_g(x, y) - K_g(x', y)| + |K_g(y, x) - K_g(y, x')| \lesssim \|g\|_{L^\infty(\mathbb{R}^n)} \frac{|x - x'|}{|x - y|^{n+1}}.$$

This implies that Π_g is a Calderón-Zygmund operator (see, for example, [24, p. 293] for the precise definition of Calderón-Zygmund operators).

Moreover, for each $H^p(\mathbb{R}^n)$ -atom a , since a has compact support, it follows that

$$\int_{\mathbb{R}^n} \Pi_g(a)(x) dx = \int_{\mathbb{R}^n} \Pi_1(a, g)(x) dx = \sum_{\substack{I, I' \in \mathcal{D} \\ |I|=|I'|}} \sum_{\lambda \in E} \langle a, \phi_I \rangle \langle g, \psi_{I'}^\lambda \rangle \int_{\mathbb{R}^n} \phi_I \psi_{I'}^\lambda(x) dx = 0,$$

which means that $\Pi_g^*(1) = 0$. By using the $T1$ theorem for Hardy spaces (see, for example, [28, Proposition 3.1]), we know that, for all $g \in L^\infty(\mathbb{R}^n)$ and $f \in H^p(\mathbb{R}^n)$ with $p \in (\frac{n}{n+1}, 1]$, $\|\Pi_1(f, g)\|_{H^p(\mathbb{R}^n)} = \|\Pi_g(f)\|_{H^p(\mathbb{R}^n)} \lesssim \|g\|_{L^\infty(\mathbb{R}^n)} \|f\|_{H^p(\mathbb{R}^n)}$, which, together with the fact that $\Lambda_\alpha(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$, implies that (3.8) holds true.

We now turn to the proof of (3.9). Let $f \in H^p(\mathbb{R}^n)$ and $g \in \dot{\Lambda}_\alpha(\mathbb{R}^n)$. Since the family $\{\psi_I^\lambda\}_{I \in \mathcal{D}, \lambda \in E}$ of wavelets is an unconditional basis of $H^p(\mathbb{R}^n)$ (see, for example, [13, Theorem 5.2]), without loss of generality, we may assume that f has a finite wavelet expansion. In this case, by Theorem 3.3, we know that $f = \sum_{l=1}^L \mu_l a_l$ has a finite atomic decomposition with the same notation as therein. Thus, by (2.6) and the fact that Π_1 is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$ (see Theorem 2.2(ii)), we find that

$$\begin{aligned} (3.10) \quad \|\Pi_1(f, g)\|_{H^1(\mathbb{R}^n)} &\lesssim \left\{ \sum_{l=1}^L |\mu_l|^p \right\}^{\frac{1}{p}} \|\Pi_1(a_l, b_l)\|_{H^1(\mathbb{R}^n)} \\ &\lesssim \|f\|_{H^p(\mathbb{R}^n)} \|a_l\|_{L^2(\mathbb{R}^n)} \|b_l\|_{L^2(\mathbb{R}^n)}, \end{aligned}$$

where

$$(3.11) \quad b_l := \sum_{\substack{I \in \mathcal{D} \\ I \subset 2mR_l}} \sum_{\lambda \in E} \langle g, \psi_I^\lambda \rangle \psi_I^\lambda,$$

with R_l being the dyadic cube related to a_l , satisfies that

$$\|b_l\|_{L^2(\mathbb{R}^n)} \lesssim \sup_{R \in \mathcal{D}} \left\{ \sum_{\substack{I \in \mathcal{D} \\ I \subset 2mR}} \sum_{\lambda \in E} |\langle g, \psi_I^\lambda \rangle|^2 \right\}^{\frac{1}{2}}.$$

By this, together with (3.10) and Theorems 3.2 and 3.3, we conclude that

$$(3.12) \quad \begin{aligned} \|\Pi_1(f, g)\|_{H^1(\mathbb{R}^n)} &\lesssim \|f\|_{H^p(\mathbb{R}^n)} \sup_{R \in \mathcal{D}} \left\{ \frac{1}{|R|^{\frac{2}{p}-1}} \sum_{\substack{I \in \mathcal{D} \\ I \subset 2mR}} \sum_{\lambda \in E} |\langle g, \psi_I^\lambda \rangle|^2 \right\}^{\frac{1}{2}} \\ &\lesssim \|f\|_{H^p(\mathbb{R}^n)} \|g\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}, \end{aligned}$$

which immediately implies that (3.9) holds true and hence completes the proof of Proposition 3.4. \square

We also have the following boundedness of the bilinear operators Π_3 and Π_4 , whose proofs are omitted due to their similarity to the proof of Proposition 3.4.

Proposition 3.5. *Let $p \in (\frac{n}{n+1}, 1)$ and $\alpha = n(\frac{1}{p} - 1)$. Then the bilinear operator Π_3 , defined as in (2.10), is bounded from $H^p(\mathbb{R}^n) \times \dot{\Lambda}_\alpha(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$ and from $H^p(\mathbb{R}^n) \times \dot{\Lambda}_\alpha(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$.*

Proposition 3.6. *Let $p \in (\frac{n}{n+1}, 1)$ and $\alpha = n(\frac{1}{p} - 1)$. Then the bilinear operator Π_4 , defined as in (2.11), is bounded from $H^p(\mathbb{R}^n) \times \dot{\Lambda}_\alpha(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.*

We point out that we need to use Theorem 2.2(i), instead of Theorem 2.2(ii), in the proof of Proposition 3.6, which justifies the space $L^1(\mathbb{R}^n)$ appearing in Proposition 3.6.

We now establish the following boundedness of the bilinear operator Π_2 . Recall that, for any $p \in (0, \infty)$, non-negative weight function w and measurable function f ,

$$(3.13) \quad \|f\|_{L_w^p(\mathbb{R}^n)} := \left[\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right]^{\frac{1}{p}}.$$

Proposition 3.7. *Let $p \in (\frac{n}{n+1}, 1)$ and $\alpha = n(\frac{1}{p} - 1)$. Then the bilinear operator Π_2 , defined as in (2.10), is bounded*

$$(3.14) \quad \text{from } H^p(\mathbb{R}^n) \times \dot{\Lambda}_\alpha(\mathbb{R}^n) \text{ to } H^p(\mathbb{R}^n)$$

and

$$(3.15) \quad \text{from } H^p(\mathbb{R}^n) \times \dot{\Lambda}_\alpha(\mathbb{R}^n) \text{ to } H_w^p(\mathbb{R}^n),$$

where w is as in (1.7) and, for all $p \in (0, 1]$, $H_w^p(\mathbb{R}^n)$ denotes the weighted Hardy space whose definition is similar to that of $H^p(\mathbb{R}^n)$ in (2.12), with the Lebesgue quasi-norm $\|\cdot\|_{L^p(\mathbb{R}^n)}$ therein replaced by the weighted Lebesgue quasi-norm $\|\cdot\|_{L_w^p(\mathbb{R}^n)}$ as in (3.13).

Proof. The proof of (3.14) is similar to that of (3.8) in Proposition 3.4, the details being omitted. We only prove (3.15). To this end, let $f \in H^p(\mathbb{R}^n)$ and $g \in \dot{\Lambda}_\alpha(\mathbb{R}^n)$. Without loss of generality, we may also assume that f has a finite wavelet expansion. In this case, since f has compact support and g is continuous, we may restrict $g \in L^2(\mathbb{R}^n)$ (see also

(3.29) below for a similar discussion). By Theorem 3.3, we know that $f = \sum_{l=1}^L \mu_l a_l$ has a finite atomic decomposition as in Theorem 3.3 with the notation same as therein. Thus,

$$(3.16) \quad \Pi_2(f, g) = \sum_{l=1}^L \mu_l \Pi_2(a_l, g).$$

Let $\{P_j\}_{j \in \mathbb{Z}}$ and $\{Q_j\}_{j \in \mathbb{Z}}$ be the orthogonal projects as in (2.7). For each $l \in \{1, \dots, L\}$, let R_l be the associated dyadic cube of the atom a_l satisfying $|R_l| = 2^{-j_l n}$ for some $j_l \in \mathbb{Z}$. Since each a_l also has a finite wavelet expansion, by an argument similar to that used in the proof of [5, Lemma 4.3], we find that there exists $j_1 \in \mathbb{N} \cap (j_l, \infty)$ such that

$$(3.17) \quad \begin{aligned} \Pi_2(a_l, g) &= \sum_{j=j_l}^{j_1-1} (Q_j a_l) (P_j g) = \sum_{j=j_l}^{j_1-1} Q_j a_l \left[P_{j_l} g + \sum_{i=j_l}^{j-1} Q_i g \right] \\ &= a_l P_{j_l} g + \sum_{j=j_l}^{j_1-1} Q_j a_l \left[\sum_{i=j_l}^{j-1} Q_i g \right]. \end{aligned}$$

Using the fact that $g = P_{j_l} g + (I - P_{j_l})g$, (3.17) and the definitions of P_j and Q_j , we know that

$$\Pi_2(a_l, g) = a_l P_{j_l} P_{j_l} g + \sum_{j=j_l}^{j_1-1} Q_j a_l \left[\sum_{i=j_l}^{j-1} Q_i (I - P_{j_l}) g \right] = a_l P_{j_l} g + \Pi_2(a_l, [I - P_{j_l}] g).$$

Moreover, by the fact that

$$(I - P_{j_l}) g = \sum_{j=j_l}^{\infty} \sum_{|I|=2^{-j n}} \sum_{\lambda \in E} \langle g, \psi_I^\lambda \rangle \psi_I^\lambda,$$

$|R_l| = 2^{-j_l n}$, (2.6) and (2.9), we further conclude that

$$\Pi_2(a_l, g) = a_l P_{j_l} g + \Pi_2 \left(a_l, \sum_{I \subset 2^m R_l} \sum_{\lambda \in E} \langle g, \psi_I^\lambda \rangle \psi_I^\lambda \right) = a_l P_{j_l} g + \Pi_2(a_l, b_l),$$

where b_l is as in (3.11).

We now claim that there exists a positive constant c , independent of g , such that, for any $l \in \{1, \dots, L\}$, $\Pi_2(a_l, g)$ can be written as

$$(3.18) \quad \Pi_2(a_l, g) = h^{(1)} + c h^{(2)} g_{R_l},$$

where $g_{R_l} := \frac{1}{|R_l|} \int_{R_l} g(x) dx$, $h^{(1)} \in H^1(\mathbb{R}^n)$ satisfies $\|h^{(1)}\|_{H^1(\mathbb{R}^n)} \lesssim \|g\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}$ and $h^{(2)}$ is an $H^p(\mathbb{R}^n)$ -atom satisfying $\|h^{(2)}\|_{H^p(\mathbb{R}^n)} \lesssim 1$.

To show the above claim, recalling that R_l is the associated dyadic cube of the atom a_l and $|R_l| = 2^{-j_l n}$, we have $a_l P_{j_l} g = \sum_{I \in \mathcal{D}, |I|=2^{-j_l n}} a_l \langle g, \phi_I \rangle \phi_I$. Observe that, for any $l \in \{1, \dots, L\}$, $I \in \mathcal{D}$ and $|I| = 2^{-j_l n}$,

$$(3.19) \quad a_l \phi_I \neq 0 \text{ if and only if } I \subset 2mR_l$$

and

$$(3.20) \quad \#\{I \in \mathcal{D} : |I| = 2^{-j_l n}, I \subset 2mR_l\} \leq (2m)^n,$$

where $\#E$ for any set E denotes the number of its elements and m is as in (2.2). By (3.19) and (3.20), we then know that the summation in $a_l P_{j_l} g$ is of finite terms with the number not more than $(2m)^n$. Moreover, for each I , we write

$$\begin{aligned} a_l \langle g, \phi_I \rangle \phi_I &= a_l \phi_I |R_l|^{-\frac{1}{2} + \frac{1}{p}} \langle g, |R_l|^{\frac{1}{2} - \frac{1}{p}} \phi_I \rangle \\ &= a_l \phi_I |R_l|^{-\frac{1}{2} + \frac{1}{p}} \left\langle g, |R_l|^{1 - \frac{1}{p}} \left(\frac{1}{|R_l|^{1/2}} \phi_I - \frac{1}{|R_l|} \chi_{R_l} \right) \right\rangle + a_l \phi_I |R_l|^{\frac{1}{2}} g_{R_l}. \end{aligned}$$

From (2.6), $\text{supp } a_l \subset R_l$ and $|R_l| = |I|$, it follows that $\text{supp}(a_l \phi_I |R_l|^{-\frac{1}{2} + \frac{1}{p}}) \subset 2mR_l$. Moreover, since a_l is an $H^p(\mathbb{R}^n)$ -atom associated with R_l , we have

$$\left\| a_l \phi_I |R_l|^{-\frac{1}{2} + \frac{1}{p}} \right\|_{L^2(\mathbb{R}^n)} \leq |R_l|^{-\frac{1}{2}}$$

and, by Theorem 3.3, we know that a_l is of the form (3.5). Thus, we conclude that $\int_{\mathbb{R}^n} a_l(x) \phi_I(x) |R_l|^{-\frac{1}{2} + \frac{1}{p}} dx = 0$, which implies that $a_l \phi_I |R_l|^{-\frac{1}{2} + \frac{1}{p}}$ is an $H^1(\mathbb{R}^n)$ -atom associated with $2mR_l$.

By a similar calculation, we find that $|R_l|^{1 - \frac{1}{p}} \left(\frac{1}{|R_l|^{1/2}} \phi_I - \frac{1}{|R_l|} \chi_{R_l} \right)$ is also an $H^p(\mathbb{R}^n)$ -atom associated with $2mR_l$, which, together with the assumption that $g \in \dot{\Lambda}_\alpha(\mathbb{R}^n)$, shows that

$$\left| \left\langle g, |R_l|^{1 - \frac{1}{p}} \left(\frac{1}{|R_l|^{1/2}} \phi_I - \frac{1}{|R_l|} \chi_{R_l} \right) \right\rangle \right| \lesssim \|g\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}.$$

Thus, we conclude that

$$(3.21) \quad \left\| a_l \phi_I |R_l|^{-\frac{1}{2} + \frac{1}{p}} \left\langle g, |R_l|^{1 - \frac{1}{p}} \left(\frac{1}{|R_l|^{1/2}} \phi_I - \frac{1}{|R_l|} \chi_{R_l} \right) \right\rangle \right\|_{H^1(\mathbb{R}^n)} \lesssim \|g\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}.$$

Now, let

$$h^{(1)} := \Pi_2(a_l, b_l) + \sum_{I \in \mathcal{D}, |I|=2^{-j_l n}} a_l \phi_I |R_l|^{-\frac{1}{2} + \frac{1}{p}} \left\langle g, |R_l|^{1 - \frac{1}{p}} \left(\frac{1}{|R_l|^{1/2}} \phi_I - \frac{1}{|R_l|} \chi_{R_l} \right) \right\rangle.$$

By the fact that Π_2 is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$ (see Theorem 2.2(ii)) and a calculation similar to (3.12), we know that

$$\|\Pi_2(a_l, b_l)\|_{H^1(\mathbb{R}^n)} \lesssim \|a_l\|_{L^2(\mathbb{R}^n)} \|b_l\|_{L^2(\mathbb{R}^n)} \lesssim \|g\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)},$$

which, combined with (3.21), implies that $h^{(1)} \in H^1(\mathbb{R}^n)$ and

$$(3.22) \quad \left\| h^{(1)} \right\|_{H^1(\mathbb{R}^n)} \lesssim \|g\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}.$$

Let $h^{(2)} := \sum_{I \in \mathcal{D}, |I|=2^{-j_l n}} \frac{1}{c} a_l \phi_I |R_l|^{\frac{1}{2}}$, where c is a positive constant such that

$$(3.23) \quad \left\| \frac{1}{c} \sum_{|I|=2^{j_l n}} a_l \phi_I |R_l|^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^n)} \leq |2mR_l|^{\frac{1}{2}-\frac{1}{p}}.$$

Using (3.19) and (3.20), we choose a positive constant c , depending only on m from (2.2), such that (3.23) holds true. This, together with the fact that a_l is an $H^p(\mathbb{R}^n)$ -atom associated with R_l of the form (3.5) and the assumption $p \in (\frac{n}{n+1}, 1)$, implies that $h^{(2)}$ is an $H^p(\mathbb{R}^n)$ -atom associated with $2mR_l$. Combining this and (3.22), we find that the claim (3.18) holds true.

With the help of the claim (3.18), we now continue the proof of (3.15). That is, we need to prove that both $h^{(1)}$ and $ch^{(2)}g_{R_l} \in H_w^p(\mathbb{R}^n)$. From (1.7) and Hölder's inequality, we deduce that, for any $h \in H^1(\mathbb{R}^n)$,

$$\begin{aligned} \|h\|_{H_w^p(\mathbb{R}^n)} &= \|h^*\|_{L_w^p(\mathbb{R}^n)} = \left\{ \int_{\mathbb{R}^n} \frac{|h^*(x)|^p}{(1+|x|)^{n(1-p)}} dx \right\}^{\frac{1}{p}} \\ &\lesssim \sum_{j=0}^{\infty} \int_{S_j(B_0)} |h^*(x)| dx \sim \|h^*\|_{L^1(\mathbb{R}^n)} \sim \|h\|_{H^1(\mathbb{R}^n)}, \end{aligned}$$

where h^* denotes the non-tangential grand maximal function defined as in (2.12), B_0 is the unit ball centered at 0 and, for any $j \in \mathbb{N}$, $S_j(B_0) := (2^{j+1}B_0) \setminus (2^jB_0)$ and $S_0(B_0) := 2B_0$. This shows that $H^1(\mathbb{R}^n) \subset H_w^p(\mathbb{R}^n)$ and hence

$$(3.24) \quad \left\| h^{(1)} \right\|_{H_w^p(\mathbb{R}^n)} \lesssim \|g\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}.$$

To estimate $\|ch^{(2)}g_{R_l}\|_{H_w^p(\mathbb{R}^n)}$, we first write

$$(3.25) \quad \left(ch^{(2)}g_{R_l} \right)^* \lesssim \left(h^{(2)} \right)^* g_{R_l} \lesssim \left(h^{(2)} \right)^* |g| + \left(h^{(2)} \right)^* |g - g_{R_l}|.$$

From the fact that $h^{(2)}$ is an $H^p(\mathbb{R}^n)$ -atom and the definition of $\dot{\Lambda}_\alpha(\mathbb{R}^n)$, we deduce that

$$\begin{aligned} (3.26) \quad &\left\| \left(h^{(2)} \right)^* |g| \right\|_{L_w^p(\mathbb{R}^n)} \\ &\lesssim \left\{ \int_{\mathbb{R}^n} \frac{|g(x) - g(0)|^p}{(1+|x|)^{n(1-p)}} \left[\left(h^{(2)} \right)^* (x) \right]^p dx \right\}^{\frac{1}{p}} + |g(0)| \left\| h^{(2)} \right\|_{H^p(\mathbb{R}^n)} \\ &\lesssim \|g\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)} + |g(0)| \sim \|g\|_{\dot{\Lambda}_\alpha^+(\mathbb{R}^n)}, \end{aligned}$$

where $\|g\|_{\dot{\Lambda}_\alpha^+(\mathbb{R}^n)} := \|g\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)} + |g(0)|$.

We now estimate the term $\|(h^{(2)})^*|g - g_{R_l}|\|_{L_w^p(\mathbb{R}^n)}$. Recall that $h^{(2)}$ is an $H^p(\mathbb{R}^n)$ -atom associated with $2mR_l$. To simplify the calculations, without loss of generality, we may assume that $R_l := \mathbb{Q}_0$ is the unit cube centered at 0 in the remainder of this proof. It is known (see, for example, [24, p. 106]) that, for all $x \in (4m\mathbb{Q}_0)^c$,

$$(h^{(2)})^*(x) \lesssim \frac{1}{(1 + |x|)^{n+1}}.$$

This, together with the definition of $\dot{\Lambda}_\alpha(\mathbb{R}^n)$ and the assumption that $p \in (\frac{n}{n+1}, 1]$, shows that

$$(3.27) \quad \left\| (h^{(2)})^* |g - g_{\mathbb{Q}_0}| \right\|_{L_w^p(\mathbb{R}^n)} \lesssim \|g\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)} \left\{ \int_{\mathbb{R}^n} \frac{1}{(1 + |x|)^{(n+1)p}} dx \right\}^{\frac{1}{p}} \lesssim \|g\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n)}.$$

Combining (3.25), (3.26) and (3.27), we conclude that $\|ch^{(2)}g_{R_l}\|_{H_w^p(\mathbb{R}^n)} \lesssim \|g\|_{\dot{\Lambda}_\alpha^+(\mathbb{R}^n)}$, where $\|\cdot\|_{\dot{\Lambda}_\alpha^+(\mathbb{R}^n)}$ is as in (3.26), which, together with (3.16), (3.18) and (3.24), implies that

$$\|\Pi_2(f, g)\|_{H_w^p(\mathbb{R}^n)} \lesssim \left(\sum_{l=1}^L |\mu_l|^p \right)^{\frac{1}{p}} \|\Pi_2(a_l, g)\|_{H_w^p(\mathbb{R}^n)} \lesssim \|f\|_{H^p(\mathbb{R}^n)} \|g\|_{\dot{\Lambda}_\alpha^+(\mathbb{R}^n)}.$$

This immediately shows that (3.15) holds true and hence finishes the proof of Proposition 3.7. \square

We now state the main result of this section.

Theorem 3.8. *Let $p \in (\frac{n}{n+1}, 1)$ and $\alpha := n(\frac{1}{p} - 1)$.*

- (i) *It holds true that there exist two bounded bilinear operators $S : H^p(\mathbb{R}^n) \times \dot{\Lambda}_\alpha(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n)$ and $T : H^p(\mathbb{R}^n) \times \dot{\Lambda}_\alpha(\mathbb{R}^n) \rightarrow H_w^p(\mathbb{R}^n)$, with w as in (1.7), such that, for all $(f, g) \in H^p(\mathbb{R}^n) \times \dot{\Lambda}_\alpha(\mathbb{R}^n)$,*

$$f \times g = S(f, g) + T(f, g).$$

- (ii) *It holds true that there exist two bounded bilinear operators $S : H^p(\mathbb{R}^n) \times \Lambda_\alpha(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n)$ and $T : H^p(\mathbb{R}^n) \times \Lambda_\alpha(\mathbb{R}^n) \rightarrow H^p(\mathbb{R}^n)$ such that, for all $(f, g) \in H^p(\mathbb{R}^n) \times \Lambda_\alpha(\mathbb{R}^n)$,*

$$f \times g = S(f, g) + T(f, g).$$

Proof. We first prove (i). For all $f \in H^p(\mathbb{R}^n)$ and $g \in \dot{\Lambda}_\alpha(\mathbb{R}^n)$, let $\{f_k\}_{k \in \mathbb{N}} \subset H^p(\mathbb{R}^n)$ have finite wavelet expansions and satisfy $\lim_{k \rightarrow \infty} f_k = f$ in $H^p(\mathbb{R}^n)$. By the definition of $f \times g$ in (3.6), we know that

$$(3.28) \quad f \times g = \lim_{k \rightarrow \infty} f_k g \quad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

where, for each $k \in \mathbb{N}$, $f_k g$ denotes the usual pointwise multiplication of f_k and g . Since f_k has a finite wavelet expansion, we know that $f_k \in L^2(\mathbb{R}^n)$ and has compact support. Let η_k be a cut-off function satisfying $\text{supp } \eta_k \subset 8m(\text{supp } f_k)$ and $\eta_k \equiv 1$ on $4m(\text{supp } f_k)$, where m is as in (2.2). It is easy to see that $f_k g = f_k(\eta_k g)$ and $\eta_k g \in L^2(\mathbb{R}^n)$. From (2.7), we deduce that $f_k g = f_k(\eta_k g) = \sum_{i=1}^4 \Pi_i(f_k, \eta_k g)$ in the sense of $L^1(\mathbb{R}^n)$ and hence of distributions.

By (2.8), (2.6) and the definition of η_k , we find that

$$\begin{aligned} (3.29) \quad \Pi_1(f_k, \eta_k g) &= \sum_{\substack{I, I' \in \mathcal{D} \\ |I|=|I'|}} \sum_{\lambda \in E} \langle f_k, \phi_I \rangle \langle \eta_k g, \psi_{I'}^\lambda \rangle \phi_I \psi_{I'}^\lambda \\ &= \sum_{\substack{I, I' \in \mathcal{D} \\ |I|=|I'|}} \sum_{\lambda \in E} \langle f_k, \phi_I \rangle \langle g, \psi_{I'}^\lambda \rangle \phi_I \psi_{I'}^\lambda = \Pi_1(f_k, g). \end{aligned}$$

Similarly, we obtain $\Pi_i(f_k, \eta_k g) = \Pi_i(f_k, g)$ for $i \in \{2, 3, 4\}$. Thus, we find that $f_k g = \sum_{i=1}^4 \Pi_i(f_k, g)$ holds true in the sense of distributions. Combining (3.28) and Propositions 3.4 through 3.7, we find that

$$f \times g = \sum_{i=1}^4 \lim_{k \rightarrow \infty} \Pi_i(f_k, g) = \Pi_4(f, g) + \left[\sum_{i=1}^3 \Pi_i(f, g) \right] =: S(f, g) + T(f, g),$$

where $S(f, g) := \Pi_4(f, g) \in L^1(\mathbb{R}^n)$ and

$$T(f, g) := \sum_{i=1}^3 \Pi_i(f, g) \in H^1(\mathbb{R}^n) + H_w^p(\mathbb{R}^n) \subset H_w^p(\mathbb{R}^n).$$

This shows that (i) holds true.

The proof of (ii) is similar to that of (i). We only need to let $S(f, g) := \Pi_4(f, g) \in L^1(\mathbb{R}^n)$ and $T(f, g) := \sum_{i=1}^3 \Pi_i(f, g) \in H^p(\mathbb{R}^n)$, which shows that (ii) holds true and hence completes the proof of Theorem 3.8. \square

Remark 3.9. (i) Recall that Bonami et al. [3] proved that, for all $p \in (0, 1)$ and $\alpha := n(\frac{1}{p} - 1)$,

$$H^p(\mathbb{R}^n) \times \dot{A}_\alpha(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + H_{w_\gamma}^p(\mathbb{R}^n)$$

with $\gamma \in (n(1-p), \infty)$ and w_γ as in (1.4). Theorem 3.8 improves the above result in two aspects when $p \in (\frac{n}{n+1}, 1)$. First, for all $\gamma \in (n(1-p), \infty)$, since, for all $x \in \mathbb{R}^n$,

$$w_\gamma(x) := \frac{1}{(1+|x|)^\gamma} < \frac{1}{(1+|x|)^{n(1-p)}} =: w(x),$$

it follows that Theorem 3.8 obtains a range space $L^1(\mathbb{R}^n) + H_w^p(\mathbb{R}^n)$ which is smaller than the range space $L^1(\mathbb{R}^n) + H_{w_\gamma}^p(\mathbb{R}^n)$ obtained in [3]. Second, the decomposition obtained in Theorem 3.8 is bilinear. Recall that, in the case $p = 1$, whether or not the product space

$H^1(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^n)$ has a bilinear decomposition consists one of the conjectures proposed in [6], which was finally solved by Bonami et al. [5].

(ii) We also point out that the bilinear decomposition obtained in Theorem 3.8(i) is sharp in the sense that the range space $L^1(\mathbb{R}^n) + H_w^p(\mathbb{R}^n)$ is almost smallest. To explain this, let A be a space smaller than $H_w^p(\mathbb{R}^n)$ and satisfy the decomposition as follows:

$$H^p(\mathbb{R}^n) \times \dot{\Lambda}_\alpha(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + A.$$

Then we have

$$(3.30) \quad \begin{aligned} (H_w^p(\mathbb{R}^n))^* \cap L^\infty(\mathbb{R}^n) &= (L^1(\mathbb{R}^n) + H_w^p(\mathbb{R}^n))^* \subset (L^1(\mathbb{R}^n) + A)^* \\ &= [A^* \cap L^\infty(\mathbb{R}^n)]. \end{aligned}$$

Moreover, For all $p \in (\frac{n}{n+1}, 1)$, $r \in (0, \infty)$ and $x \in \mathbb{R}^n$, let $\psi(x, r) := [\frac{r}{1+|x|}]^{n(\frac{1}{p}-1)}$ and w be as in (1.7). Define the BMO type space $\text{BMO}_\psi(\mathbb{R}^n)$ by setting

$$\begin{aligned} \text{BMO}_\psi(\mathbb{R}^n) &:= \left\{ f \in L_{\text{loc}}^1(\mathbb{R}^n) : \|f\|_{\text{BMO}_\psi(\mathbb{R}^n)} \right. \\ &\quad \left. := \sup_{\substack{x \in \mathbb{R}^n \\ r \in (0, \infty)}} \left[\frac{1}{\psi(x, r)|B(x, r)|} \int_{B(x, r)} |f(y) - f_{B(x, r)}| dy \right] < \infty \right\}, \end{aligned}$$

where $f_{B(x, r)} := \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y) dy$. By [19, Theorem 3.2] and some elementary technical calculations, we know that

$$[(H_w^p(\mathbb{R}^n))^* \cap L^\infty(\mathbb{R}^n)] = [\text{BMO}_\psi(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)],$$

which coincides with the pointwise multiplier class of $\dot{\Lambda}_{n(\frac{1}{p}-1)}(\mathbb{R}^n)$ (see [23, Theorem 1]).

By the fact that the product of $H^p(\mathbb{R}^n) \times \dot{\Lambda}_\alpha(\mathbb{R}^n)$ is well defined for all pointwise multipliers of $\dot{\Lambda}_{n(\frac{1}{p}-1)}(\mathbb{R}^n)$ as in (3.7), if the duality argument is possible (which may not be the case since the function spaces dealt here may not be Banach spaces), we conclude that

$$[A^* \cap L^\infty(\mathbb{R}^n)] \subset [\text{BMO}_\psi(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)] = [(H_w^p(\mathbb{R}^n))^* \cap L^\infty(\mathbb{R}^n)],$$

which, together with (3.30), implies that $[A^* \cap L^\infty(\mathbb{R}^n)] = [(H_w^p(\mathbb{R}^n))^* \cap L^\infty(\mathbb{R}^n)]$. In this sense, we say that $H_w^p(\mathbb{R}^n)$ and A coincide with each other and hence the range space $L^1(\mathbb{R}^n) + H_w^p(\mathbb{R}^n)$ is almost smallest.

We point out that the bilinear decomposition of Theorem 3.8(ii) is also almost smallest in the same meaning as in Theorem 3.8(i), in view of the fact that the pointwise multiplier class of $\Lambda_{n(\frac{1}{p}-1)}(\mathbb{R}^n)$ coincides with the space $\Lambda_{n(\frac{1}{p}-1)}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ which is just $(L^1(\mathbb{R}^n) + H^p(\mathbb{R}^n))^*$ (see [23, Theorem 3]), the details being omitted.

4 Applications to the Div-Curl Lemma

In this section, we apply the bilinear decompositions of the products of Hardy spaces and their duals, obtained in Section 3, to the study of the div-curl lemma at the endpoint case $q = \infty$. To this end, we first introduce some notation. For all $p \in (0, \infty)$, let

$$(4.1) \quad H^p(\mathbb{R}^n; \mathbb{R}^n) := \{\mathbf{F} := (F_1, \dots, F_n) : \text{ for all } i \in \{1, \dots, n\}, F_i \in H^p(\mathbb{R}^n)\}$$

and, for any $\mathbf{F} \in H^p(\mathbb{R}^n; \mathbb{R}^n)$, let

$$\|\mathbf{F}\|_{H^p(\mathbb{R}^n; \mathbb{R}^n)} := \left[\sum_{i=1}^n \|F_i\|_{H^p(\mathbb{R}^n)}^2 \right]^{\frac{1}{2}}.$$

The vector-valued spaces $h^p(\mathbb{R}^n; \mathbb{R}^n)$, $\dot{\Lambda}_\alpha(\mathbb{R}^n; \mathbb{R}^n)$ and $\text{bmo}(\mathbb{R}^n; \mathbb{R}^n)$ are defined similarly, the details being omitted.

The following theorem is the main result of this section.

Theorem 4.1. *For all $p \in (\frac{n}{n+1}, 1)$ and $\alpha := n(\frac{1}{p} - 1)$, let $\mathbf{F} \in H^p(\mathbb{R}^n; \mathbb{R}^n)$ with $\text{curl } \mathbf{F} \equiv 0$ and $\mathbf{G} \in \dot{\Lambda}_\alpha(\mathbb{R}^n; \mathbb{R}^n)$ with $\text{div } \mathbf{G} \equiv 0$ (both of the equalities hold true in the sense of distributions). Then $\mathbf{F} \cdot \mathbf{G} \in H_w^p(\mathbb{R}^n)$, where w is as in (1.7).*

Proof. Let $\mathbf{F} \in H^p(\mathbb{R}^n; \mathbb{R}^n)$ with $\text{curl } \mathbf{F} \equiv 0$ and $\mathbf{G} \in \dot{\Lambda}_\alpha(\mathbb{R}^n; \mathbb{R}^n)$ with $\text{div } \mathbf{G} \equiv 0$. By Theorem 3.8, we know that

$$\mathbf{F} \cdot \mathbf{G} = \sum_{i=1}^n F_i \times G_i = \sum_{i=1}^n S(F_i, G_i) + \sum_{i=1}^n T(F_i, G_i) =: A(\mathbf{F}, \mathbf{G}) + B(\mathbf{F}, \mathbf{G}).$$

From Theorem 3.8(i), it immediately follows that $B(\mathbf{F}, \mathbf{G}) \in H_w^p(\mathbb{R}^n)$ and

$$\|B(\mathbf{F}, \mathbf{G})\|_{H_w^p(\mathbb{R}^n)} \lesssim \|\mathbf{F}\|_{H^p(\mathbb{R}^n; \mathbb{R}^n)} \|\mathbf{G}\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n; \mathbb{R}^n)}.$$

Thus, to finish the proof of (i), it suffices to show $A(\mathbf{F}, \mathbf{G}) \in H_w^p(\mathbb{R}^n)$. To this end, we only need to prove that $A(\mathbf{F}, \mathbf{G}) \in H^1(\mathbb{R}^n)$ because of the inclusion $H^1(\mathbb{R}^n) \subset H_w^p(\mathbb{R}^n)$. Since $L^2(\mathbb{R}^n; \mathbb{R}^n)$ is dense in $H^p(\mathbb{R}^n; \mathbb{R}^n)$, without loss of generality, we may assume that $\mathbf{F} \in H^p(\mathbb{R}^n; \mathbb{R}^n) \cap L^2(\mathbb{R}^n; \mathbb{R}^n)$. Using the Helmholtz decomposition (see, for example, [6, Section 4] for more details), we find that there exists

$$f := - \sum_{i=1}^n R_i(F_i) \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$$

such that $\mathbf{F} = \nabla(-\Delta)^{-1/2}f$, where, for all $i \in \{1, \dots, n\}$, R_i denotes the i -th Riesz transform. Moreover, since $\text{div } \mathbf{G} \equiv 0$, it follows that $\sum_{i=1}^n R_i(G_i) \equiv 0$. Thus, we can write

$$A(\mathbf{F}, \mathbf{G}) = \sum_{i=1}^n S(F_i, G_i) = \sum_{i=1}^n [S(R_i(f), G_i) + S(f, R_i(G_i))].$$

Using (2.11) and the fact that R_i is a Calderón-Zygmund operator with odd kernel, we further find that, for each $i \in \{1, \dots, n\}$,

$$\begin{aligned}
& S(R_i(f), G_i) + S(f, R_i(G_i)) \\
&= \sum_{I, I' \in \mathcal{D}} \sum_{\lambda, \lambda' \in E} \langle f, \psi_I^\lambda \rangle \langle G_i, \psi_{I'}^{\lambda'} \rangle \langle R_i \psi_I^\lambda, \psi_{I'}^{\lambda'} \rangle \left(\psi_{I'}^{\lambda'} \right)^2 \\
&\quad + \sum_{I, I' \in \mathcal{D}} \sum_{\lambda, \lambda' \in E} \langle f, \psi_I^\lambda \rangle \langle G_i, \psi_{I'}^{\lambda'} \rangle \langle \psi_I^\lambda, R_i \psi_{I'}^{\lambda'} \rangle \left(\psi_I^\lambda \right)^2 \\
&= \sum_{I, I' \in \mathcal{D}} \sum_{\lambda, \lambda' \in E} \langle f, \psi_I^\lambda \rangle \langle G_i, \psi_{I'}^{\lambda'} \rangle \langle R_i \psi_I^\lambda, \psi_{I'}^{\lambda'} \rangle \left[\left(\psi_{I'}^{\lambda'} \right)^2 - \left(\psi_I^\lambda \right)^2 \right],
\end{aligned}$$

which, together with some calculations similar to those used in the proof of [5, Lemma 6.1], implies that

$$\|S(R_i(f), G_i) + S(f, R_i(G_i))\|_{H^1(\mathbb{R}^n)} \lesssim \sum_{I, I' \in \mathcal{D}} \sum_{\lambda, \lambda' \in E} \left| \langle f, \psi_I^\lambda \rangle \right| \left| \langle G_i, \psi_{I'}^{\lambda'} \rangle \right| p_\delta(I, I'),$$

where, for all $\delta \in (0, \frac{1}{2}]$, $|I| = 2^{-j^n}$ and $|I'| = 2^{-j'^n}$ with centers at x_I , respectively, $x_{I'}$,

$$p_\delta(I, I') := 2^{-|j-j'|(\delta+n/2)} \left(\frac{2^{-j} + 2^{-j'}}{2^{-j} + 2^{-j'} + |x_I - x_{I'}|} \right)^{n+\delta}.$$

This shows that the coefficient matrix of \mathbf{A} is almost diagonal (see [12, Theorem 3.3] for more details), which, combined with a dual argument on the sequence space, the boundedness of the Riesz transform $\nabla(-\Delta)^{-1/2}$ on $H^p(\mathbb{R}^n)$ (see, for example, [24, p. 115, Theorem 4]) and Theorem 3.2, implies that

$$\begin{aligned}
& \|S(R_i(f), G_i) + S(f, R_i(G_i))\|_{H^1(\mathbb{R}^n)} \\
&\lesssim \sum_{I' \in \mathcal{D}} \sum_{\lambda' \in E} \left[\sum_{I \in \mathcal{D}} \sum_{\lambda \in E} \left| \langle f, \psi_I^\lambda \rangle \right| p_\delta(I, I') \right] \left| \langle G_i, \psi_{I'}^{\lambda'} \rangle \right| \\
&\lesssim \left\| \left\{ \sum_{I \in \mathcal{D}} \sum_{\lambda \in E} \left| \langle f, \psi_I^\lambda \rangle \right| p_\delta(I, I') \right\}_{I' \in \mathcal{D}, \lambda' \in E} \right\|_{\dot{f}_{p,2}^0(\mathbb{R}^n)} \left\| \left\{ \langle G_i, \psi_{I'}^{\lambda'} \rangle \right\}_{I' \in \mathcal{D}, \lambda' \in E} \right\|_{C_p(\mathbb{R}^n)} \\
&\lesssim \left\| \left\{ \langle f, \psi_I^\lambda \rangle \right\}_{I \in \mathcal{D}, \lambda \in E} \right\|_{\dot{f}_{p,2}^0(\mathbb{R}^n)} \left\| \left\{ \langle G_i, \psi_{I'}^{\lambda'} \rangle \right\}_{I' \in \mathcal{D}, \lambda' \in E} \right\|_{C_p(\mathbb{R}^n)} \\
&\lesssim \|\mathbf{F}\|_{H^p(\mathbb{R}^n; \mathbb{R}^n)} \|\mathbf{G}\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n; \mathbb{R}^n)},
\end{aligned}$$

where $\|\cdot\|_{\dot{f}_{p,2}^0(\mathbb{R}^n)}$ and $\|\cdot\|_{C_p(\mathbb{R}^n)}$ is defined as in (3.3), respectively, Theorem 3.2. This shows that $\mathbf{A}(\mathbf{F}, \mathbf{G}) \in H^1(\mathbb{R}^n)$ and

$$\|\mathbf{A}(\mathbf{F}, \mathbf{G})\|_{H^1(\mathbb{R}^n)} \lesssim \|\mathbf{F}\|_{H^p(\mathbb{R}^n; \mathbb{R}^n)} \|\mathbf{G}\|_{\dot{\Lambda}_\alpha(\mathbb{R}^n; \mathbb{R}^n)},$$

which completes the proof of Theorem 4.1. \square

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